Let $(\Omega, \mathcal{F}, P)$ be a probability space. All random variables or stochastic processes below will be on this probability space.

1. (a) Let $\{B_t\}$ be a standard Brownian motion. Prove that, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} B_{2n} = 0.$$

(b) Let $\{X_t\}_{t\in[0,\infty)}$ be a stochastic process. Show that the following are equivalent.
(i) $\{X_t\}$ is a Brownian motion.
(ii) $\{X_t\}$ is a centred Gaussian process with $Cov[X_s, X_t] = s \wedge t$, $\forall s, t \geq 0$.
(c) Let $\{B_t\}$ be a Brownian motion. Using part (b) or otherwise, show that $\{\frac{1}{2}B_{4t}\}$ is also a Brownian motion.

Solution:
(a)

$$\frac{1}{n}B_{2n} = \frac{(B_{2n} - B_{2n-2}) + (B_{2n-2} - B_{2n-4}) + \cdots + (B_2 - B_0)}{n}.$$

Now, $(B_{2i} - B_{2i-2}) \sim N(0, 2)$ for each $i = 1, \cdots, n$. And for each i, $(B_{2i} - B_{2i-2})$ are i.i.d. random variables. Therefore by the law of large numbers $\frac{1}{n}B_{2n} \to 0$, as $n \to \infty$.

(b) The covariance function determines the finite-dimensional distributions of a centered Gaussian process since a multidimensional normal distribution is determined by the vector of expectations and by the covariance matrix. Therefore X is characterized by (ii). Hence, it is enough to show that, for Brownian motion X, we have $Cov[X_s, X_t] = \min(s, t)$. This is indeed true since for $t > s$, the random variables X_s and $X_t - X_s$ are independent; hence

$$Cov[X_s, X_t] = Cov[X_s, X_t - X_s] + Cov[X_s, X_s] = Var[X_s] = s.$$

(c) Say $X_t := \frac{1}{2}B_{4t}$. From (b), for $t > s$

$$Cov[X_s, X_t] = Var[X_s] = Var[\frac{1}{2}B_{4s}] = \frac{1}{4} \cdot 4s = s.$$

And, also $\mathbb{E}[\frac{1}{2}B_{4t}] = 0$, Since $\mathbb{E}[B_t] = 0$. Therefore $\{\frac{1}{2}B_{4t}\}$ is also a Brownian motion.

$\square$

2. Let (B_t) be a standard one dimensional Brownian motion.
(a) Let X be an $N(0, 1)$-distributed r.v., which is independent of $\{B_t\}$. For any $t \in [0, 1]$ show that

$$\mathbb{P}(\sqrt{1-t}|X| \leq |B_t|) = \frac{2}{\pi} \arcsin(\sqrt{t}).$$

(b) Let $\tau_b := \inf\{s > 0 : B_s = b\}$. Show that for $b > 0$,

$$\mathbb{E}(e^{\lambda \tau_b}) = e^{-b\sqrt{2\lambda}}.$$

(c) Show that for any $a > 0$, $t > 0$

$$\mathbb{P}\{\sup\{B_s, 0 \le s \le t\} > a\} = 2\mathbb{P}\{B_t > a\}.$$

Solution: (a) Let $\tilde{B}$ be an another independent Brownian motion. By the reflection principle,

$$\begin{aligned}
&\mathbb{P}[B_s \neq 0 \ \forall s \in [t,1]] \\
&= \int_{-\infty}^{\infty} \mathbb{P}[B_s \neq 0 \ \forall s \in [t,1] | B_t = a] \mathbb{P}[B_t \in da] \\
&= \int_{-\infty}^{\infty} \mathbb{P}_{|a|}[\tilde{B}_s > 0 \ \forall s \in [0, 1-t]] \mathbb{P}[B_t \in da] \\
&= \int_{-\infty}^{\infty} \mathbb{P}_0[\tilde{B}_{1-t} \le |a|] \mathbb{P}[B_t \in da] \\
&= \mathbb{P}[|\tilde{B}_{1-t}| \le |B_t|].
\end{aligned}$$

Define

$$X := \frac{B_t}{\sqrt{t}}.$$

Now, if X and Y are independent and $N(0,1)$-distributed, then

$$(B_t, \tilde{B}_{1-t}) \overset{\mathcal{D}}{=} (\sqrt{t}X, \sqrt{1-t}Y).$$

Hence

$$\begin{aligned}
&\mathbb{P}[\sqrt{1-t}|Y| \le \sqrt{t}|X|] \\
&= \mathbb{P}[Y^2 \le t(X^2 + Y^2)] \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-(x^2+y^2)/2} \mathbb{1}_{\{y^2 \le t(x^2+y^2)\}} \\
&= \frac{1}{2\pi} \int_{0}^{\infty} r \ dr \ e^{-r^2/2} \int_{0}^{2\pi} d\varphi \mathbb{1}_{\{\sin(\varphi)^2 \le t\}} \quad \text{[by polar coordinates]} \\
&= \frac{2}{\pi} \arcsin(\sqrt{t}).
\end{aligned}$$

Therefore,

$$\mathbb{P}(\sqrt{1-t}|X| \le |B_t|) = \frac{2}{\pi} \arcsin(\sqrt{t}).$$

(b) By Itô's formula we can show that $(\exp(\sigma B_t - \frac{\sigma^2}{2} t))_{t \ge 0}$ is a martingale. Let us denote, $M_t :=$

$\exp(\sigma B_t - \frac{\sigma^2}{2}t)$. Therefore, by martingale property,

$$\begin{aligned}
&\mathbb{E}[M_{\tau_b}] = \mathbb{E}[M_0] \\
\Rightarrow &\mathbb{E}[\exp(\sigma b - \frac{\sigma^2}{2}\tau_b)] = 1 \\
\Rightarrow &\exp(\sigma b)\mathbb{E}[\exp(-\frac{\sigma^2}{2}\tau_b)] = 1 \\
\Rightarrow &\mathbb{E}[\exp(-\frac{\sigma^2}{2}\tau_b)] = \frac{1}{\exp(\sigma b)}.
\end{aligned}$$

Now, choose $\sigma = \sqrt{2\lambda}$. Therefore

$$\mathbb{E}[\exp(-\lambda\tau_b)] = \frac{1}{\exp(b\sqrt{2\lambda})}.$$

(c) If B is a Brownian motion and if $K \neq 0$, then $(K^{-1}B_{K^2t})_{t\geq 0}$ is also a Brownian motion. Without loss of generality, we may assume $t = 1$. Let $\tau := \inf\{s \geq 0 : B_s \geq a\} \wedge 1$. By symmetry, we have $\mathbb{P}_a[B_{1-\tau} > a] = \frac{1}{2}$ if $\tau < 1$; hence

$$\begin{aligned}
\mathbb{P}[B_1 > a] &= \mathbb{P}[B_1 > a|\tau < 1]\mathbb{P}[\tau < 1] \\
&= \mathbb{P}_a[B_{1-\tau} > a]\mathbb{P}[\tau < 1] \\
&= \frac{1}{2}\mathbb{P}[\tau < 1].
\end{aligned}$$

Therefore the result follows.

□

3. Let (B_t) be any one dimensional Brownian motion, $(\mathcal{F}_t)$ its natural filtration and τ a finite stopping time.
(a) Show that $(B_{t+\tau} - B_\tau)_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{F}_\tau$.
(b) Using part a) or otherwise show that the strong Markov property holds at τ.

Solution: (a) We first show our statement for the stopping times τ_n with discretely approximate τ from above, $\tau_n = (m+1)2^{-n}$ if $m2^{-n} \leq \tau < (m+1)2^{-n}$. Write $B_k = \{B_k(t) : t \geq 0\}$ for the Brownian motion defined by $B_k(t) = B(t + k/2^n) - B(k/2^n)$, and $B_* = \{B_*(t) : t \geq 0\}$ for the process defined by $B_*(t) = B(t + \tau_n) - B(\tau_n)$. Suppose that $E \in \mathcal{F}^+_{\tau_n}$. Then, for every event $\{B_* \in A\}$, we have

$$\begin{aligned}
\mathbb{P}(\{B_* \in A\} \cap E) &= \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\} \cap E \cap \{\tau_n = k2^{-n}\}) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\}\mathbb{P}(E \cap \{\tau_n = k2^{-n}\}),
\end{aligned}$$

using that $\{B_k \in A\}$ is independent of $E \cap \{\tau_n = k2^{-n}\} \in \mathcal{F}^+_{k2^{-n}}$. Again, $\mathbb{P}\{B_k \in A\} = \mathbb{P}\{B \in A\}$ does not depend on k, hence

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\}\mathbb{P}(E \cap \{\tau_n = k2^{-n}\}) &= \mathbb{P}\{B \in A\} \sum_{k=0}^{\infty} \mathbb{P}(E \cap \{\tau_n = k2^{-n}\}) \\
&= \mathbb{P}\{B \in A\}\mathbb{P}(E),
\end{aligned}$$

which shows that B_* is a Brownian motion and independent of E, hence of $\mathcal{F}^+_{\tau_n}$.
It remains to generalise this to general stopping times τ. As $\tau_n \downarrow \tau$ we have that $\{B(s+\tau_n)-B(\tau_n) : s \geq 0\}$ is a Brownian motion independent of $\mathcal{F}^+_{\tau_n} \supset \mathcal{F}^+_{\tau}$. Hence the increments

$$B(s+t+\tau) - B(t+\tau) = \lim_{n\to\infty} [B(s+t+\tau_n) - B(t+\tau_n)]$$

of the process $\{B(r+\tau) - B(\tau) : r \geq 0\}$ are independent and normally distributed with mean zero and variance s. As the process is obviously almost surely continuous, it's a Brownian motion. Moreover all increments, $B(s+t+\tau) - B(t+\tau) = \lim_{n\to\infty}[B(s+t+\tau_n) - B(t+\tau_n)]$, and hence the process itself, are independent of $\mathcal{F}^+_\tau$.

(b) Let $\mathbb{P}_x$ denote the probability measure s.t. $B = (B_t)_{t\geq 0}$ is a Brownian motion started at $x \in \mathbb{R}$, i.e. the process $(B_t - x)_{t\geq 0}$ is a standard Brownian motion.
We have to show that, for every bounded measurable $F : \mathbb{R}^{[0,\infty)} \to \mathbb{R}$ and $x \in \mathbb{R}$,

$$\mathbb{E}_x[F((B_{t+\tau})_{t\geq 0})|\mathcal{F}_\tau] = \mathbb{E}_{B_\tau}[F(B)]. \tag{1}$$

It is enough to consider continuous bounded functions F that depend on only finitely many coordinates $t_1, \cdots, t_N$ since these functions determine the distribution of $(B_{t+\tau})_{t\geq 0}$. Hence, let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be continuous and bounded $F(B) = f(B_{t_1}, \cdots, B_{t_N})$. The map $x \mapsto \mathbb{E}_x[F(B)] = \mathbb{E}_0[f(B_{t_1}+x, \cdots, B_{t_N}+x)]$ is continuous and bounded. Now let $\tau^n := 2^{-n}\lfloor 2^n\tau + 1\rfloor$ for $n \in \mathbb{N}$. Then τ^n is a stopping time and $\tau^n \downarrow \tau$; hence $B_{\tau^n} \xrightarrow{n\to\infty} B_\tau$ a.s.. Now every Markov process with countable time set (here all positive rational linear combinations of $1, t_1, \cdots, t_N$) is a strong Markov process. Hence,

$$\begin{aligned}\mathbb{E}_x[F((B_{\tau^n+t})_{t\geq 0})|\mathcal{F}_{\tau^n}] &= \mathbb{E}_x[f(B_{\tau^n+t_1}, \cdots, B_{\tau^n+t_N})|\mathcal{F}_{\tau^n}] \\ &= \mathbb{E}_{B_{\tau^n}}[f(B_{t_1}, \cdots, B_{t_N})] \\ &\xrightarrow{n\to\infty} \mathbb{E}_{B_\tau}[f(B_{t_1}, \cdots, B_{t_N})] = \mathbb{E}_{B_\tau}[F(B)].\end{aligned} \tag{2}$$

As B is right continuous, we have $F((B_{\tau^n+t})_{t\geq 0}) \xrightarrow{n\to\infty} F((B_{\tau+t})_{t\geq 0})$ almost surely and in L^1 and thus

$$\begin{aligned}&\mathbb{E}[|\mathbb{E}_x[F((B_{\tau^n+t})_{t\geq 0})|\mathcal{F}_{\tau^n}] - \mathbb{E}_x[F((B_{\tau+t})_{t\geq 0})|\mathcal{F}_{\tau^n}]|] \\ &\leq \mathbb{E}_x[|F((B_{\tau^n+t})_{t\geq 0}) - F((B_{\tau+t})_{t\geq 0})|] \xrightarrow{n\to\infty} 0.\end{aligned} \tag{3}$$

Furthermore,

$$\mathcal{F}_{\tau^n} \downarrow \mathcal{F}_{\tau+} := \bigcap_{\sigma>\tau \text{ is a stopping time}} \mathcal{F}_\sigma \supset \mathcal{F}_\tau.$$

By (2) and (3), we get

$$\begin{aligned}\mathbb{E}_{B_\tau}[F(B)] &= \lim_{n\to\infty} \mathbb{E}_x[F((B_{\tau^n+t})_{t\geq 0})|\mathcal{F}_{\tau^n}] \\ &= \lim_{n\to\infty} \mathbb{E}_x[F((B_{\tau+t})_{t\geq 0})|\mathcal{F}_{\tau^n}] \\ &= \mathbb{E}_x[F((B_{\tau+t})_{t\geq 0})|\mathcal{F}_{\tau+}].\end{aligned}$$

The L.H.S. is $\mathcal{F}_\tau$- measurable. The tower property of conditional expectation thus yields (1).

$\square$